ON THE FAMOUS PROBLEM OF CALCULUS OF VARIATION: A BRACHISTOCHRONE PROBLEM

A. G. Patil
Department of Mathematics, Kisan Arts Commerce and Science College, Parola, Dist Jalgaon. (M.S.)

ABSTRACT:

The Second Order non-linear, two-point boundary value problem, governing the famous problem of Calculus of variation namely, brachistochrone problem has been studied. The finite Lie Group transformation technique is applied to transform the boundary value problem into initial value problem. The advantage of present non-iterative solution over conventional analytical solution is also discussed.

KEY WORDS: Brachistochrone problem, Boundary value problem, Initial value problem, Group theoretic method.

1. Introduction:

John Bernoulli (1667-1748), a younger brother of famous Swiss mathematician, James Bernoulli (1654-1705) has posed a novel problem as a challenge “to the most acute mathematicians of the entire world” in the year 1996. According to John Bernoulli, suppose a wire is bent in to the shape of an arbitrary curve and asked which curve among the infinitely many possibilities will give the shortest possible time of descent. This curve is called the brachistochrone. The word brachistochrone is a Greek word, which means shortest (brachistos) time (chrons), mathematically above problem can be stated as follow:

Suppose two nails are driven at random in to a wall, and let upper nail be connected to the lower by a wire in the shape of smooth curve. What is the shape of the wire down which a bead will slide (without friction) under the influence of gravity so as to pass from the upper nail to the lower in least possible time?

When John Bernoulli has posed above challengeous problem to the world of mathematicians, the world famous mathematician Sir Isaac Newton has left Cambridge for London to become Warden, and soon Master of the Mint. During the reminder of his long life, he entered a little in to the society and even began to enjoy his unique position at the pinnacle of scientific fame. These changes in interest and surroundings did not yet caused decreases in his unrivaled intellectual powers. One late afternoon at the end of the hard day at the Mint, he learned about the change of Swiss mathematician. He has immediately summoned his resources and solved brachistochrone problem that evening before going to the bed. His solution was published anonymously, and when John Bernoulli show it, he wryly commented " I recognize the lion by his claw".

Our purpose in this paper is to study brachistochrone problem by two different approaches: analytical approach and numerical approach.

2.Problem formulation

For the problem stated in above article, let us denote fixed points by A (a, y_a) and B (b, y_b). Then our aim is to find shape of the wire for which transit time can become minimum when frictionless bead slides down the curved wire joining A and B. In fact, this problem is nothing but generalization of usual minimax problems of differential calculus.

Let S denote the arc length along the curve measured from the initial point (a,y_a). We have total time of descent T for the curve y= y (x) containing points A(a,y_a) and B (b, y_b) is given by:

$$T = \int_0^T dt = \int_0^{S_1} \frac{dt}{ds} ds = \int_0^{S_1} \frac{1}{v} ds \tag{1}$$

Where, S_1 is the total arc length of the curve and v is the velocity of the mass.

Since, $ds=(1+y^{\bullet 2})^{1/2}dx$ we have,

$$T = \int_a^b \frac{(1+y^{\bullet 2})^{1/2}}{v} dx \qquad (2)$$

Where, $y^{\bullet}$ denotes derivative with respect to x.

Now, we know that energy is conserved through the motion of beads, that is,

(Kinetic energy at t > 0) + (Potential energy at t > 0)

= (Kinetic energy at t = 0) + (Potential energy at t = 0)

$$\Rightarrow \tfrac{1}{2} mv^2 + mgy = 0 + mgy_a$$

$$\Rightarrow v = \sqrt{2g}\,(y_a - y(x))^{1/2} \qquad (3)$$

Substituting this value of v in equation (2) we get,

$$T(y) = (2g)^{-1/2} \int_a^b \left(\frac{1+y^{\bullet 2}}{y_a - y}\right)^{1/2} dx \qquad (4)$$

This is the quantity which we wish to minimize correctly by choosing a curve y(x) subject to the boundary conditions $y(a) = y_a$ and $y(b) = y_b$.

The quantity in equation (4) is of the form $A(y) = \int_a^b L(x,y,y^{\bullet})\, dx$ (5)

Where A(y) is a functional, that is a quantity whose value is a number but argument is a function y(x). The function L is given by

$$L(x, y, y^{\bullet}) = (2g)^{-1/2}\left(\frac{1+y^{\bullet 2}}{y_a - y}\right)^{1/2} \qquad (6)$$

is call Lagrangian.

Now, applying Euler- Lagrangian equation.

$$\frac{dL_{y^{\bullet}}}{dx} - L_y = 0 \qquad (7)$$

to equation (6) we get,

$$2(y - y_a)\frac{d^2y}{dx^2} + \left(1 + \left(\frac{dy}{dx}\right)^2\right) = 0 \qquad (8)$$

Now it is more convenient if we transform co-ordinate from $y - y_a = z$ then equation (8) will transform into following two point non linear boundary value problem:

$$2z\frac{d^2z}{dx^2} + \left(1 + \left(\frac{dy}{dx}\right)^2\right) = 0 \qquad (9)$$

With boundary conditions:

$$z(0) = Z_0 : \quad \frac{dz(x_T)}{dx} = 0 \qquad (10)$$

Where, $z_0 = y_0 - y_a$, $x = x_T$ is a terminal line on which point B may lie any where in such a way that resistance, dy of the bead sliding on the wire, at the point B is zero.

That is $\frac{dy}{dx} = 0 \Rightarrow \frac{dz}{dx} = 0$ at $x = x_T$

Now, the analytical solution of equation (6) with boundary conditions $y(a)= y_a$ and $y(b)=y_b$ is obvious and it is discussed in several classical books on calculus of variations.[Refer: COURANT et. al.(1953), LOGAN (1987),GELFAND et al (1963)]

Most recently, according to DRESNER,(1999), the solution of (6) will be following parametric equations of cycloid.

$$x = a + (2c^2)^{-1} [2\theta - \sin 2\theta] \quad (11)$$

$$y = y_a - (2c^2)^{-1} [1 - \cos 2\theta] \quad (12)$$

It is surprising to note that here in contrast to the problem of finding the curve of shortest length between two points, it is not clear that the cycloids just given by equations (11) and (12) minimize the functional given by equation (5) and hence we need further investigations. Thus here we have physical situation for which numerical method need to be preferred dispite an analytical solution being available.

Introducing following transformations in to the boundary value problem given by equations (9) and (10).

$$\xi = 1 - \frac{x}{x_T} \; ; \quad g = \frac{y}{x_T} \quad (13)$$

We get,

$$2g \frac{d^2g}{d\xi^2} + 1 + \left(\frac{dg}{d\xi}\right)^2 = 0 \quad (14)$$

With the boundary conditions:

$$g(1) = S \quad \text{and} \quad \frac{dg(0)}{d\xi} = 0 \quad (15)$$

Where $S = \frac{z_0}{x_T}$

To solve above second order non-linear boundary value problem numerically. We first reduce it in to initial value problem for this we have to find missing initial condition at the point $\xi = 0$

To transform boundary value problem (14)-(15), we introduce following one parameter-scaling group of transformation [Refer TIMOL et al (1986),(2007), (2009), (2010)]

$$\xi = \lambda^{\alpha}\,\bar{\xi} \qquad g = \lambda^{\beta}\,\bar{g} \tag{16}$$

The equation (14) will remain invariant, that is independent of group parameter λ for $\alpha=\beta$ and thus equation (14) will transformed in to:

$$2\bar{g}\,\frac{d^2\bar{g}}{d\bar{\xi}^2} + 1 + \left(\frac{d\bar{g}}{d\bar{\xi}}\right)^2 = 0 \tag{17}$$

With the second boundary condition, (15), becomes

$$\frac{d\bar{g}(0)}{d\xi} = 0 \tag{18}$$

Next, missing initial condition can be obtained by selecting

$$g(0) = \lambda \tag{19}$$

Under transformation (16) equation (19) becomes, $\lambda^{\beta}\,\bar{g}(0) = \lambda$

Which gives $\beta = 1$ and thus $g(0) = 1$ (20)

Further $\beta = 1$ gives $\alpha = 1$ as $\alpha = \beta$.

Finally from the first initial condition,(15) and transformation(16), we have

$$\bar{g}(1) = S/\lambda \quad \text{and} \quad \bar{\xi} = 1/\lambda \tag{21}$$

Eliminating λ from equation (21),

we get $$S = \bar{g} / \bar{\xi} \tag{22}$$

Thus, we have initial value problem given by equation (17) with initial conditions (18) and (19). Hence, we can find non-iterative solution of equation (14) for a given value of S.

The Taylor's series solution of equation (17)-(19) gives,

$$\bar{g}(\bar{\xi}) = 1 - \bar{\xi}^2/4 + 0(\bar{\xi}^3) \tag{23}$$

Neglecting higher order terms and combining equations (22) and (23) we get quadratic equation in $\bar{\xi}$ which finally yields.

$$\bar{\xi} = 2\,[\sqrt{s^2+1} - s\,] \tag{24}$$

Results and discussion:

First calculate $\bar{\xi}$ for a given value of S then using equation (22) finds corresponding value of $\bar{g}$. Using this values of $\bar{\xi}$ and $\bar{g}$ calculate λ using equation (21). Finally, the solution of original equation, equation (14), can be obtained through the definition of transformation, equation (16). These solutions are included in Table-1. Thus we have found optimal path, which satisfies all the constraints.

Table

(For s = 1.00, $\bar{\xi}$ = 0.83 and A = 1.2071)

$\bar{\xi}$	$\bar{g}$	ξ	**G**
0.00	1.0000	0.00	1.2071
0.10	0.9975	0.12	1.2040
0.20	0.9900	0.24	1.1950
0.30	0.9775	0.36	1.7990
0.40	0.9600	0.48	1.1588
0.50	0.9375	0.60	1.1316
0.60	0.9100	0.72	1.0984
0.70	0.8775	0.84	1.0592
0.83	0.8277	1.00	0.9991

Conclusion:

The non-iterative solution of the brachistochrone problem is obtained. The group theoretic method is used to transform original two-point boundary value problem, describing the path of beads sliding on a wire, into initial value problem. The conventional analytical solution of the brachistochrone problem, which gives parametric equations of cycloid, fails to give the curve of shortest length between two points. Where as present non-iterative solution gives such optimal path between two points in a gravitational field.

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